On the independence of axioms in BL and MTL☆

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Abstract

We prove that the axiom expressing that the multiplicative conjunction of two formulae implies the first one of them is redundant in the standard Hilbert-style calculi of Hájek's basic logic BL and Esteva and Godo's monoidal t-norm based logic MTL. This proof does not use the axiom expressing that multiplicative conjunction is commutative, which is already known to be redundant. Therefore both of these axioms are simultaneously redundant. We also show that all the other axioms are independent of each other.

Keywords: Non-classical logics, Basic fuzzy logic (BL), Monoidal t-norm based logic (MTL), Hilbert-style calculi, Independence of axioms

1. Introduction

Hájek's basic logic BL, developed in [5], and Esteva and Godo's monoidal t-norm based logic MTL, see [4], are prominent examples of formal systems of mathematical fuzzy logic. They arise as the logic of continuous and the logic of left-continuous t-norms respectively. The standard Hilbert-style calculus of BL comes from Hájek, cf. [5]. Esteva and Godo [4] adapted his system for MTL, replacing one axiom with three new axioms.

As both calculi are almost identical, Cintula [2] proved that the axiom (A3), which expresses that multiplicative conjunction is commutative, is provable from the other axioms in both BL and MTL. Lehmke [6] proved that the axiom (A2), which expresses that the multiplicative conjunction of two formulae implies the first one of them, is also provable from the other axioms.

These two results, however, do not prove the redundancy of both these axioms simultaneously, because each of the proofs uses the other axiom. We overcome this problem by presenting a new proof of the axiom (A2), which does not use the axiom (A3). Moreover, we show by semantic arguments that all the other axioms are independent of each other.

All these results were obtained with the essential help of a computer—the proofs of the axiom (A2) were found by the E prover 0.999-001 [7] and semantic counterexamples by the finite model finder Paradox 2.3 [3]. We used a standard technique based on the encoding of propositional formulae as terms in the classical FOL. No further exposition of these techniques is presented, because the purpose of this paper is to present the metamathematical results in adapted and readable form; not to develop the techniques. A reader interested in the technique itself can consult papers by Larry Wos, see [8].

This paper is organised as follows. In the following section some basic definitions are given. Section 3 contains the proofs of the axiom (A2) in MTL and BL. In Section 4 we show that all the remaining axioms are independent of each other.

This paper is based on [1], the author's contribution to the proceedings of a local student conference.

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2. Preliminaries

In this paper we study two propositional fuzzy logics. We use standard terminology from the theory of logical calculi, see e.g. [5]. The language of these two logics consists of implication ($\to$), multiplicative conjunction (&) and a constant for falsity ($\bot$). Moreover, the language of MTL contains additive conjunction ($\wedge$), which is a defined connective in BL.

As we are only interested in two particular presentations of BL and MTL, we define both the logics by these presentations. Consequently, in this paper, we understand a logic as a formal system. Moreover, we are only interested in Hilbert-style calculi. It is worth pointing out that we consider Hilbert-style calculi in the form where the substitution rule is implicit. Therefore, strictly speaking, these systems consist of axiom schemata, but for simplicity we talk just about axioms.

We define a proof (a derivation) of a formula φ as a finite sequence of formulae such that each of its members is an axiom or follows from some of the preceding members by modus (ponendo) ponens, and the last member of the sequence is φ.

Definition 1 ([5])**.** We define the basic logic BL as a Hilbert-style calculus with the following formulae as axioms:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(A2) $(\varphi \,\&\, \psi) \to \varphi$,
(A3) $(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi)$,
(A4) $(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$,
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$,
(A5b) $((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$,
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,
(A7) $\bot \to \varphi$.

The only deduction rule of BL is modus ponens

(MP) $\varphi, \varphi \to \psi \,/\, \psi$.

As our purpose is to prove the axiom (A2) without using the axiom (A3), we define a new system in which we prove this statement.

Definition 2. The logic BL^- is obtained by omitting the axioms (A2) and (A3) from BL.

The monoidal t-norm based logic MTL is obtained by weakening the properties of the additive conjunction which is no longer an abbreviation for $\varphi \,\&\, (\varphi \to \psi)$. Therefore we define the additive conjunction directly by three new axioms.

Definition 3 ([4])**.** We obtain the monoidal t-norm based logic MTL by replacing the axiom (A4) in BL with the following three axioms:

(A4a) $(\varphi \,\&\, (\varphi \to \psi)) \to (\varphi \wedge \psi)$,
(A4b) $(\varphi \wedge \psi) \to \varphi$,
(A4c) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$.

Definition 4. The logic MTL^- is obtained by omitting the axioms (A2) and (A3) from MTL.

3. The provability of the axiom (A2)

In this section we prove the axiom (A2) from the other axioms, without the use of the axiom (A3). Note that we use an easy consequence of the axiom (A1), namely for any χ, if $\varphi \to \psi$ is provable then $(\psi \to \chi) \to (\varphi \to \chi)$ is also provable. Let us also note that we do not refer explicitly to the use of modus ponens as it is our only deduction rule.

It suffices to construct a derivation of weakening $\varphi \to (\psi \to \varphi)$, because from this formula we immediately obtain (A2) by the residuation axiom (A5a).

3.1. MTL$^-$

First, we prove (A2) in MTL$^-$. Surprisingly, the proof is shorter than in BL$^-$, because the axioms (A4a) and (A4b) shorten the proof significantly. These formulae are provable in BL, since it is a stronger logic than MTL, but we do not know whether it is also possible to prove them in BL$^-$. Therefore we have to prove (A2) separately for MTL$^-$ and BL$^-$.

Lemma 1. *The following formulae are provable in* MTL$^-$*:*

(a) $(\varphi \,\&\, (\varphi \to \psi)) \to \varphi$,
(b) $(((\varphi \to \psi) \to \varphi) \to \chi) \to (\varphi \to \chi)$,
(c) $\varphi \to (\varphi \to \varphi)$,
(d) $\varphi \to (\psi \to \varphi)$.

Proof. We prove the first formula using (A4a) on (A1) which gives us $((\varphi \wedge \psi) \to \chi) \to ((\varphi \,\&\, (\varphi \to \psi)) \to \chi)$. To prove (a), we now apply (A4b).

The second formula can be proved using (A1) on

$$\varphi \to ((\varphi \to \psi) \to \varphi), \tag{1}$$

which is an easy consequence of (a) used on (A5b).

The proof of the third formula starts by proving $((\varphi \to ((\varphi \to \psi) \to \varphi)) \to \varphi) \to (\varphi \to ((\varphi \to \psi) \to \varphi))$, which is just (1) used on itself, where $\varphi = \varphi \to ((\varphi \to \psi) \to \varphi)$ and $\psi = \varphi$. Then this formula and (b) give us $\varphi \to (\varphi \to ((\varphi \to \psi) \to \varphi))$, hence we prove $((\varphi \to ((\varphi \to \psi) \to \varphi)) \to \varphi) \to (\varphi \to \varphi)$ by (A1) and hence $\varphi \to (\varphi \to \varphi)$ by (b).

Finally, we prove the last formula. We start by proving $((\varphi \to (\psi \to (\varphi \& \psi))) \to \varphi) \to (((\varphi \& \psi) \to (\varphi \& \psi)) \to \varphi)$, which is (A5b) used on (A1) and hence $\varphi \to (((\varphi \,\&\, \psi) \to (\varphi \,\&\, \psi)) \to \varphi)$ by (b). Now we can substitute $\varphi \to (\varphi \to \varphi)$ for φ in the previous formula and by the application of (c) we obtain $(((\varphi \to (\varphi \to \varphi))\&\psi) \to ((\varphi \to (\varphi \to \varphi))\&\psi)) \to (\varphi \to (\varphi \to \varphi))$, hence $((\varphi \to (\varphi \to \varphi)) \,\&\, \psi) \to (\varphi \to (\varphi \to \varphi))$, because $((\varphi \to \varphi) \to \psi) \to (\varphi \to \psi)$ is provable by using (c) on (A1). Now using (A5b) and (c) we prove $\psi \to (\varphi \to (\varphi \to \varphi))$, hence $((\varphi \to (\varphi \to \varphi)) \to \varphi) \to (\psi \to \varphi)$ by (A1) and finally $\varphi \to (\psi \to \varphi)$ by (b). □

An immediate consequence of this lemma and the axiom (A5a) is the following theorem.

Theorem 2. *The axiom (A2) is provable in* MTL$^-$.

It is worth pointing out that we did not use the axioms (A4c), (A6), and (A7). Contrarily, all the other axioms are necessary, which can be demonstrated by methods used in Section 4.

As we know from [2] that the axiom (A3) is provable from the other axioms in MTL, we immediately obtain the following corollary.

Corollary 3. *The axioms (A2) and (A3) are simultaneously redundant in* MTL.

3.2. BL$^-$

Second, we prove (A2) in BL$^-$. As we have already pointed out, this proof is longer than the proof in MTL$^-$. We would like to note that although we use the axioms (A6) and (A7) to prove (A2) in BL$^-$, these axioms are not necessary but they shorten the proof significantly. All the other axioms are necessary.

Lemma 4. *The following formulae are provable in* BL$^-$*:*

(a) $\varphi \to \varphi$,
(b) $(\varphi \,\&\, (\varphi \to \bot)) \to \psi$,
(c) $(\varphi \,\&\, \psi) \to \psi$,
(d) $\varphi \to (\psi \to \varphi)$.

Proof. We start with the proof of (a). First, we prove $((\varphi \to \varphi) \& \varphi) \to \varphi$ by a double application of $((\varphi \to \varphi) \to (\varphi \to \varphi)) \to (((\varphi \to \varphi) \& \varphi) \to \varphi)$, which is an instance of (A5a), on a suitable instance of (A6). Hence $(\varphi \to \varphi) \to (\varphi \to \varphi)$ by (A5b). Again by a double application of the previous formula on a suitable instance of (A6) we prove $\varphi \to \varphi$.

Next, we prove (b). First, we prove $((\varphi \to \psi) \& \varphi) \to \psi$ by (a) and (A5a). Now we prove

$$(\psi \to \chi) \to (((\varphi \to \psi) \& \varphi) \to \chi) \tag{2}$$

by (A1). Then $(\bot \& \psi) \to \varphi$ is proved by (A7) and (A5a). Hence $((\varphi \to (\bot \& \psi)) \& \varphi) \to \chi$ by (2) and $(\varphi \to (\bot \& \psi)) \to (\varphi \to \chi)$ by (A5b). Finally, we prove (b) by an instance of (A4) applied on the previous formula where $\varphi = \varphi \& (\varphi \to \bot)$, $\psi = \bot \to \varphi$, and $\chi = \psi$.

We proceed to proving the third formula. We obtain $\varphi \to ((\varphi \to \bot) \to \psi)$ by (b) and (A5b), hence $(((\varphi \to \bot) \to \psi) \to \chi) \to (\varphi \to \chi)$ by (A1). From $((\varphi \to \bot) \to \psi) \to (\bot \to \psi)$, which is an immediate consequence of (A7) and (A1), and an instance of the previous formula we prove $\varphi \to (\bot \to \psi)$. Since $\varphi \to (\psi \to (\varphi \& \psi))$ is provable by (a) and (A5b), we can easily prove $(\psi \to \psi) \& ((\psi \to \psi) \to (\bot \to \varphi))$, hence

$$(\bot \to \varphi) \& ((\bot \to \varphi) \to (\psi \to \psi)) \tag{3}$$

by (A4). Moreover, using $(\chi \& \varphi) \to (\bot \to \varphi)$ we prove $\chi \to (\varphi \to (\bot \to \varphi))$ by (A5b). Hence $((\varphi \to (\bot \to \varphi)) \to \xi) \to (\chi \to \xi)$ by (A1) and using a suitable instance of (A1) on this formula, for $\xi = ((\bot \to \varphi) \to \eta) \to (\varphi \to \eta)$, we prove $\chi \to (((\bot \to \varphi) \to \eta) \to (\varphi \to \eta))$, hence $(\chi \& ((\bot \to \varphi) \to \eta)) \to (\varphi \to \eta)$ by (A5a). Hence $\varphi \to (\psi \to \psi)$ by (3). Now (A5a) gives us (c).

To prove the last formula, we begin with $((\psi \& \chi) \to \chi) \to (((\varphi \to (\psi \& \chi)) \& \varphi) \to \chi)$, which is an instance of (2). Hence $(\varphi \to (\psi \& \chi)) \to (\varphi \to \chi)$ by (c) and (A5b). Now we take an instance of the previous formula such that its antecedent is (A4), hence $(\varphi \& (\varphi \to \psi)) \to (\psi \to \varphi)$. From this we prove

$$(((\varphi \to \psi) \to (\psi \to \varphi)) \to \chi) \to (\varphi \to \chi) \tag{4}$$

by (A5b) and (A1). Now $(\psi \to \varphi) \to (\psi \to \varphi)$ and (A6) give us $((\varphi \to \psi) \to (\psi \to \varphi)) \to (\psi \to \varphi)$. We complete the proof using this formula on (4). ☐

Again, the immediate consequence of this lemma and the axiom (A5a) is the following theorem, which together with the provability of the axiom (A3) in BL [2] implies the redundancy of both axioms.

Theorem 5. *The axiom (A2) is provable in* BL^-.

Corollary 6. *The axioms (A2) and (A3) are simultaneously redundant in* BL.

4. The independence of the other axioms

In this section we show that (A2) and (A3) are the only axioms that can be proved from the other axioms in BL and MTL. In other words, each of the other axioms is independent of the remaining ones. Since the standard argument used to show that some axiom is independent of the other axioms is based on semantics, we present some necessary notions. Let us remark that we adapt the standard notions for our very restricted purposes. A matrix $\mathcal{M}$ is a pair $\langle A, \{1\}\rangle$ where A is an algebra with the signature $(\to, \&, \wedge, \bot)$ and 1 is the only designated element of A. The carrier of A is a subset of $\{0, a, b, c, 1\}$. We can present such a matrix simply by giving tables for all connectives. Unless otherwise stated, $\bot$ is interpreted as 0. An $\mathcal{M}$-valuation e is a mapping from formulae to the elements of A such that e commutes over connectives. We say that some axiom φ is valid in $\mathcal{M}$ if for any $\mathcal{M}$-valuation e the equation $e(\varphi) = 1$ holds. Otherwise we say that the axiom φ is invalid. We say that modus ponens is valid in $\mathcal{M}$ if $e(\varphi) = e(\varphi \to \psi) = 1$ implies $e(\psi) = 1$ for any $\mathcal{M}$-valuation e and formulae φ, ψ.

The whole argument reads as follows: Let us have a matrix in which all axioms and modus ponens are valid. Therefore all provable formulae are valid in the matrix, thus any formula not valid in the matrix is not provable.

We present a matrix giving tables for all connectives. In most cases, the table can be the same for both conjunctions. Moreover, only the logic-specific axioms (A4), (A4a), (A4b), and (A4c) have to be treated separately for BL and MTL, all the other axioms can be studied for both logics at once. Let us also stress that the axioms (A2) and (A3)

&, ∧	0	a	b	1
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
1	0	a	0	1

→	0	a	b	1
0	1	1	1	1
a	1	1	a	1
b	1	1	1	1
1	a	a	a	1

Table 1: Truth tables for (A1)

&	0	a	b	1
0	0	0	0	0
a	0	0	b	b
b	0	b	0	b
1	0	b	b	1

∧	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
1	0	a	a	1

→	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	b	1	1	1
1	0	b	a	1

Table 2: Truth tables for (A5a)

are valid in all presented matrices. Therefore the independence results hold even in the (full) standard presentations of BL and MTL.

We do not present proofs that all the remaining axioms and modus ponens are valid in a given matrix. Some cases are trivial, but some other cases need exhaustive checking, which would make this paper much longer. An interested reader can check all the details, e.g. by a computer program. On the other hand, we give valuations which show that a particular axiom is invalid in a matrix and therefore independent. Let us note that an $\mathcal{M}$-valuation e is uniquely determined by its values on propositional variables. Therefore we define it by its values on φ, ψ, and χ.

Let us start with the axioms common to both logics.

4.1. The axiom (A1)

The matrix $\mathcal{M}$ is defined by Table 1. The axiom (A1) is invalid in this structure for a valuation e such that $e(\varphi) = a$, $e(\psi) = 0$, and $e(\chi) = b$.

4.2. The axiom (A5a)

The axiom (A5a) is invalid in the structure defined by Table 2, as shown by a valuation e satisfying $e(\varphi) = b$, $e(\psi) = a$, and $e(\chi) = 0$.

4.3. The axiom (A5b)

The structure $\mathcal{M}$ with only two elements $\{0, 1\}$ is sufficient to show the independence of the axiom (A5b). Both conjunctions have the value 0 for any pair of arguments and the implication is defined as in the classical logic. The axiom (A5b) is invalid for a valuation e such that $e(\varphi) = e(\psi) = 1$ and $e(\chi) = 0$.

4.4. The axiom (A6)

The independence of the axiom (A6), which represents the prelinearity condition, is well-known. Moreover, the system obtained by omitting the axiom (A6) from MTL is known under many names e.g. Ono's $\mathrm{FL_{ew}}$ or Höhle's Monoidal Logic ML.

Nevertheless, we present our standard semantic argument. The matrix is based on Table 3 and represents the well-known five-element Heyting-algebra with two incomparable elements $a, b < c$. The axiom (A6) is invalid as proved by a valuation e, where φ, ψ, and χ have values a, b, and c.

&, ∧	0	*a*	*b*	*c*	1
0	0	0	0	0	0
a	0	*a*	0	*a*	*a*
b	0	0	*b*	*b*	*b*
c	0	*a*	*b*	*c*	*c*
1	0	*a*	*b*	*c*	1

→	0	*a*	*b*	*c*	1
0	1	1	1	1	1
a	*b*	1	*b*	1	1
b	*a*	*a*	1	1	1
c	0	*a*	*b*	1	1
1	0	*a*	*b*	*c*	1

Table 3: Truth tables for (A6)

&	0	*a*	*b*	1
0	0	0	0	0
a	0	0	0	*a*
b	0	0	0	*b*
1	0	*a*	*b*	1

→	0	*a*	*b*	1
0	1	1	1	1
a	*b*	1	1	1
b	*b*	*b*	1	1
1	0	*a*	*b*	1

Table 4: Truth tables for (A4)

4.5. The axiom (A7)

It is clear that the axiom (A7) is independent of the other axioms, because it is the only axiom that contains the symbol $\bot$—it is enough to interpret $\bot$ as 1 and all the connectives classically. In such model, the axiom (A7) is invalid and all the other axioms and modus ponens are evidently valid.

All the remaining axioms are logic-specific. We start with the axiom specific to BL and then proceed to the three axioms specific to MTL.

4.6. The axiom (A4)

We do not present a table for the additive conjunction, because $\varphi \wedge \psi$ is just an abbreviation for $\varphi \,\&\, (\varphi \to \psi)$. The axiom (A4) is invalid in the matrix given by Table 4 as shown by a valuation e with $e(\varphi) = a$ and $e(\psi) = b$.

4.7. The axioms (A4a), (A4b), and (A4c)

We present three matrices with two elements $\{0, 1\}$. These matrices differ only in how the additive conjunction is defined. The multiplicative conjunction and the implication are defined as the conjunction and the implication for classical logic. Therefore the axioms (A1)–(A3) and (A5a)–(A7) are trivially valid in these matrices.

For (A4a) we define the additive conjunction in such a way that it has value 0 for all pairs of values. The axiom (A4a) is invalid as proved by e such that $e(\varphi) = e(\psi) = 1$.

For (A4b) we define the additive conjunction in such a way that it has value 0 only for $0 \wedge 0$ and value 1 for all other pairs of values. The axiom (A4b) is invalid as proved by e such that $e(\varphi) = 0$ and $e(\psi) = 1$.

For (A4c) we define the additive conjunction in such a way that its value equals the value of its first member ($x \wedge y = x$). The axiom (A4c) is invalid as proved by valuation e such that $e(\varphi) = 1$ and $e(\psi) = 0$.

If we combine all the results of this section, we immediately obtain the following two corollaries.

Corollary 7. *All the axioms but (A2) and (A3) are independent of each other in* BL.

Corollary 8. *All the axioms but (A2) and (A3) are independent of each other in* MTL.

5. Summary

We proved that the axiom (A2) is provable from the other axioms in BL and MTL without the use of the axiom (A3), which was shown in [2] to be provable from the remaining axioms. Therefore we proved that both of these axioms are simultaneously redundant. Moreover, we demonstrated that no other axiom has this property even in the (full) standard presentations of BL and MTL with (A2) and (A3) and therefore all the other axioms are independent of each other.

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